ONLINE APPENDIX

F. Other Proofs of Results in Appendix D

Proof of Lemma D.1: Observe that for $s \in (\underline{t}, t]$, μ_s^h faces an outflow rate of $(\rho+\lambda)\mu_s^h$ and an inflow rate of 1, where ρ is the risk of detection and λ the risk of transition to the low state. That is

$$\frac{\partial \mu_s^h}{\partial s} = 1 - (\rho+\lambda)\mu_s^h.$$

Solving this simple differential equation with the initial condition $\mu_0^h = 0$ leads to the result. $\square$

Proof of Lemma D.2: First, observe

$$\lim_{t\to\infty} \left(w^h(t) - e^{-(\rho+r)t}\underline{p}\right) = \frac{x^h - x^l}{\rho+r+\lambda} - \frac{\rho\overline{p} - x^l}{\rho+r} = \frac{x^h - x^l}{\rho+r+\lambda} - \Delta_l - \underline{p} \leq -\underline{p} \tag{F.1}$$

where the inequality follows since $\theta \in \Theta^*$, and is strict whenever $\theta \in (\Theta^*)^{\mathrm{o}}$. Further,

$$w^h(0) - e^{-(\rho+r)0}\underline{p} = -\underline{p} \tag{F.2}$$

Observe next that

$$\frac{\partial}{\partial t}\left(w^h(t) - e^{-(\rho+r)t}\underline{p}\right) = e^{-(\rho+r)t}\left((x^h - x^l)e^{-\lambda t} + (\rho+r)\underline{p} - (\rho\overline{p} - x^l)\right) \tag{F.3}$$

Since $\theta \in \Theta^*$, equation (F.3) is strictly positive at 0 and crosses zero exactly once. Combining (F.2), (F.3) and the strict version of (F.1) implies that for $\theta \in (\Theta^*)^{\mathrm{o}}$, equation (5) has a unique strictly positive solution $t(p) > 0$, which is strictly increasing. If instead $\theta \in \partial\Theta^*$, then $\mathcal{P} = \{-\underline{p}\}$ and equation (5) has a unique non-zero solution at $t(p) = \infty$.

Finally, since $t(p)$ is strictly increasing, $\inf_{p\in\mathcal{P}} t(p) = t(\underline{p}) > 0$. $\square$

Proof of Lemma D.3: Let $f = \rho + r + \lambda$. Plugging definitions, we get:

$$v^0(t) + e^{-rt}\frac{v^0(\underline{t})}{1-e^{-r\underline{t}}} = \frac{1}{\rho+\lambda}\left(\frac{1-e^{-ft}}{f} + e^{-rt}\frac{1-e^{-f\underline{t}}}{f(1-e^{-r\underline{t}})}\right)$$

Differentiate the term in parentheses on the right-hand side with respect to t to get:

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{1-e^{-ft}}{f} + \frac{1-e^{-f\underline{t}}}{f(1-e^{-r\underline{t}})}\right) &= e^{-ft} - \frac{r}{f}e^{-rt}\frac{1-e^{-f\underline{t}}}{1-e^{-r\underline{t}}} \\ &\leq e^{-ft} - \frac{r}{f}e^{-rt}\frac{1-e^{-ft}}{1-e^{-rt}}\end{aligned}$$

$$= \frac{1-e^{-ft}}{f}\left[\frac{fe^{-ft}}{1-e^{-ft}} - \frac{re^{-rt}}{1-e^{-rt}}\right]$$
$$= \frac{1-e^{-ft}}{f}\left[\phi(f) - \phi(r)\right]$$

where $\phi(a) \equiv \frac{ae^{-at}}{1-e^{-at}}$ and the second line follows for any $t \geq \underline{t}$ because $\frac{1-e^{-ft}}{1-e^{-rt}}$ is decreasing in t. To see that $\phi(f) - \phi(r) \leq 0$, note that $f > r$ and

$$\frac{\partial}{\partial a}\phi(a) = \frac{e^{at}\left(1-(at+e^{-at})\right)}{(e^{at}-1)^2}$$
$$\leq 0$$

for any $at \geq 0$ since $z + e^{-z} > 1$ for any $z \geq 0$. This concludes the first part of the lemma.

For the second part of the lemma, observe that, by Lemma D.2, $t(p)$ defined by equation (5) is increasing in p. Applying the first part of this lemma then leads to the result. □

G. Proof of Theorem A.1

I prove here that the optimal deterministic policy delivers the regulator a larger value than any Poisson random policy.

Proof of Theorem A.1: Let $f \equiv \rho + r + \lambda$. In a γ-Poisson policy, the recommendation $a_t(x^h) = \mathbf{1}_{p_t=0}$ is incentive compatible if and only if

$$\mathbb{E}_\gamma\left(w^h(t)\right) \leq 0 \tag{G.1}$$

where $\mathbb{E}_\gamma$ denotes the expectation operator for t distributed as an exponential distribution with rate parameter γ. Recalling the relationship between $\mathbf{V}$ and v from Lemma 4, the result follows if I can show that any γ-Poisson policy satisfying equation (G.1) is such that

$$\sup_{t_0\geq 0} v(t_0) + e^{-rt_0}\frac{-\mathbb{E}_\gamma\left(v(t)\right)}{\mathbb{E}_\gamma\left(1-e^{-rt}\right)} < \sup_{t_0\geq 0} v(t_0) + e^{-rt_0}\frac{-v(t(\underline{p}))}{1-e^{-rt(\underline{p})}}$$

where $t(\underline{p})$ is the unique strictly positive solution to equation (5) at $p = \underline{p}$ (when $\theta \in \Theta^*$), the right-hand side is an affine transformation of the regulator's value from the optimal deterministic policy, and the left-hand side is the regulator's value from a γ-Poisson policy. Since the choice of t_0 has the same domain in both problems, and a solution $t_0 \in \mathbb{R}_+$ exists for both problems, it is sufficient to show that

$$\frac{\mathbb{E}_\gamma\left(v(t)\right)}{\mathbb{E}_\gamma\left(1-e^{-rt}\right)} < \frac{v(t(\underline{p}))}{1-e^{-rt(\underline{p})}} \tag{G.2}$$

First, I will manipulate inequality (G.1). From the definition of $t(\underline{p})$, we have $\frac{x^h v(t(\underline{p}))}{1-e^{-(\rho+r)t(\underline{p})}} = \frac{\rho\overline{p}}{\rho+r}$. Recalling that $w(t) = x^h v(t) - \frac{\rho\overline{p}}{\rho+r}(1-e^{-(\rho+r)t})$, we have

$$\mathbb{E}_\gamma\left(x^h v(t) - \frac{\rho\overline{p}}{\rho+r}(1-e^{-(\rho+r)t})\right) \leq 0 \tag{G.3}$$

$$\iff \mathbb{E}_\gamma\left(x^h v(t) + e^{-(\rho+r)t}\frac{x^h v(t(\underline{p}))}{1-e^{-(\rho+r)t(\underline{p})}}\right) \leq \frac{x^h v(t(\underline{p}))}{1-e^{-rt(\underline{p})}} \tag{G.4}$$

Plugging in for $v(t)$ and rearranging inequality (G.2), we find

$$\frac{\mathbb{E}_\gamma\left(v(t)\right)}{\mathbb{E}_\gamma\left(1-e^{-rt}\right)} < \frac{v(t(\underline{p}))}{1-e^{-rt(\underline{p})}} \tag{G.5}$$

$$\iff \mathbb{E}_\gamma\left(1-e^{-ft} + e^{-rt}\frac{1-e^{-ft(\underline{p})}}{1-e^{-rt(\underline{p})}}\right) \leq \frac{1-e^{-ft(\underline{p})}}{1-e^{-rt(\underline{p})}} \tag{G.6}$$

After canceling constants, observe that inequalities (G.2) and (G.1) are special cases of the inequality:

$$\mathbb{E}_\gamma\left[(1-e^{-ft})(1-(e^{-ft^*})^a) + (e^{-ft})^a(1-e^{-ft^*})\right] - (1-e^{-ft^*}) \leq 0 \tag{$C^\gamma(a)$}$$

where $a = \frac{r}{\rho+r+\lambda}$ for the regulator and $a = \frac{\rho+r}{\rho+r+\lambda}$ for the agent and I've multiplied both sides by $(1-(e^{-(\rho+r+\lambda)t(\underline{p})})^a)$. Denote the left-hand side by $h(a; z, \gamma)$. As in Theorem 2, the crucial step is the following:

$$\text{if } h(a; z, \gamma) \leq 0 \textit{ at some } \overline{a} \in (0,1), \text{ then } h(a; z, \gamma) < 0 \text{ for each } 0 < a \leq \overline{a}. \tag{C^γ}$$

With this the proof will be concluded. Let $z = e^{-ft(\underline{p})}$. Integrating with respect to t, the inequality becomes

$$(1-z^a)(1-\frac{\gamma}{\gamma+f}) + \frac{\gamma}{\gamma+fa}(1-z) - (1-z) \leq 0$$

Rather than show Property (C^γ) for h, I will show it for $\tilde{h} = h(\gamma + (\rho+r+\lambda)a)$, from which the property for h can be recovered (since for $a \in [0,1]$, $sgn(h) = sgn(\tilde{h})$). Computing $\tilde{h}$, we see that:

$$\tilde{h} = (1-z^a)(\gamma+fa) - \gamma\frac{\gamma+fa}{\gamma+f}(1-z^a) + \gamma(1-z) - (1-z)(\gamma+fa)$$

I claim that if $\frac{\partial^2\tilde{h}}{\partial a^2}$ has at most one 0, then Property C^γ will be verified and the proof will be complete. To see this, observe that as $a \downarrow -\infty$, $\tilde{h} \uparrow \infty$. Observe also that $\tilde{h}(0) = \tilde{h}(1) = 0$. To violate the property, there must exist points $a_1 < 0 < a_2 < a_3 < 1$ such that $\tilde{h}(a_1), \tilde{h}(a_2) \geq 0$, $\tilde{h}(a_3) \leq 0$, while $\tilde{h}(0) = \tilde{h}(1) = 0$. This requires $\frac{\partial^2\tilde{h}}{\partial\tilde{h}^2}$ to pass through 0 *at least twice*. So, I proceed to show that $\frac{\partial^2 \tilde{h}}{\partial a^2}$ has at most one 0.

Twice differentiating $\tilde{h}$ leads to:

$$\frac{\partial^2 \tilde{h}}{\partial a^2} = -\left(\frac{f^2}{\gamma + f}\right) z^a ln(z) \left[2 + ln(z)a\right]$$

Since $z > 0$, this has exactly one zero at $a = -\frac{2}{ln(z)}$. So, I conclude that $\frac{\partial^2 \tilde{h}}{\partial a^2}$ crosses 0 at most once so that Property C^γ holds, and the conclusion follows. $\square$

H. Proofs for Section 6

Proof of Proposition 3: Suppose $\gamma < \rho$. Lemma 3 proceeds in exactly the same way. The statement of Lemma 4 must now be altered so that rather than applying a discount of $e^{-(\gamma+r)t}$ the regulator applies a discount of $e^{-\gamma t}$, but is otherwise identical. To see this, fix some $t \geq 0$ and policy $(\mathbf{p}, \mathbf{a}) \in \mathcal{M}^0$. Recall that, by the definition of $\mathcal{M}^0$, there exists a sequence $\mathbf{t} = (t_i)_{i \in \mathbb{N}}$ such that $a_t(x^h) = 1$ if and only if $t \in \{t_i\}_{i \in \mathbb{N}}$. Then, integrating and rearranging yields,

$$\begin{aligned} V(\mathbf{p}, \mathbf{a}) &= -\sum_{i=0}^{\infty} \int_0^{\mathbf{t}_i(\mathbf{p},\mathbf{a})} e^{-(\gamma+r)t} \left(\int_t^{\mathbf{t}_i(\mathbf{p},\mathbf{a})} e^{-(\rho+r+\lambda)(s-t)} ds \right) dt \\ &= -\frac{1}{(\gamma+r)(\rho+\lambda-\gamma)} + \frac{1}{\rho+\lambda-\gamma} \sum_{i=0}^{\infty} e^{-(r+\gamma)s_{i-1}(\mathbf{p},\mathbf{a})} \frac{1 - e^{-(\rho+r+\lambda)t_i(\mathbf{p},\mathbf{a})}}{\rho+r+\lambda} \end{aligned}$$

So a version of Lemma 4 holds using the recursive equation

$$\mathbf{V}(p) = \begin{cases} \sup\limits_{t,p'} \quad v(t) + e^{-(r+\gamma)t} \mathbf{V}(p') \\ \text{subject to} \\ w^h(t) - e^{-(\rho+r)t} p' \leq -p \\ p' \in [\underline{p}, \frac{\rho \overline{p}}{\rho+r} - \frac{x^h}{\rho+r+\lambda}] \end{cases}$$

As long as $\gamma < \rho$, Proposition 2 can be derived with exactly the same steps, and the result of Theorem 2 follows. $\square$

Proof of Proposition 4: Let V^{cont} be the regulator's value associated to the policy described in the proposition. To prove the result, expand the set $\mathcal{M}^0$ to include the policies $(\mathbf{p}, \mathbf{a})$ such that there exists $\epsilon > 0$ such that:

- If t is not a limit-point of the set $\{t | a_t(x^h) = 1\}$, then $a_{[t-\epsilon, t+\epsilon]}(x^h) = 0$.
- If t is a limit-point of $\{t | a_t(x^h) = 1\}$, then $t \in I \subset \{t | a_t(x^h) = 1\}$ where I is an interval.

- If t' is a right-endpoint of an interval $[t, t']$ such that $a_{[t,t']}(x^h) = 1$, then $a_{[t',t'+\epsilon}(x^h) = 0$.

The first requirement says that reporting times must be separated by an ϵ if they are not in an interval. The second requirements say that if t is a limit-point of the set of high type reporting times, then it is an element of an interval. The third says that if t is an endpoint of such an interval, there is a uniform lower bound (across time) for how long the policy must wait until another reporting time.

Denote by $\mathcal{M}^{cont}$ the set of policies that satisfy these requirements. Recall from Lemma 3, that $V^* = \sup_{(\mathbf{p},\mathbf{a})\in\mathcal{M}^0} V(\mathbf{p}, \mathbf{a})$, where M^0 was the set of policies that satisfied the first requirement above and had no limit points in reporting time. Since, $M^0 \subset M^{cont}$, then $V^* = \sup_{(\mathbf{p},\mathbf{a})\in\mathcal{M}^{cont}} V(\mathbf{p}, \mathbf{a})$. To prove the result, it is then sufficient to show that

$$V^{cont} = \sup_{(\mathbf{p},\mathbf{a})\in\mathcal{M}^{cont}} V(\mathbf{p}, \mathbf{a}).$$

Recursive Representation. I first write the regulator's problem recursively in almost the same way as in the main text. The only difference now is that the regulator can also choose an interval of time during which to set $a_t(x^h) = 1$. The regulator's problem is written recursively:

- A *decision node* of the regulator is any reporting time of the high type *that is not an interior point of an interval* of reporting times

- The *choice* of the regulator is now either

  - *next reporting time* and *penalty at next reporting time* or
  - *length of an interval* on which to continuously induce reporting by high types, as well as the penalty offered at the end of this interval

  Let $d = 0$ indicate the regulator is choosing the former and $d = 1$ the latter.

- The *state* of the regulator is the reporting penalty that he must offer to the agent immediately

- The *constraint* of the regulator is

  - if $d = 0$, the one-shot incentive compatibility condition we have seen
  - if $d = 1$,
    - penalty at the end of the interval > penalty at the beginning i.e. the state
    - a maximum length of the interval, to be defined below.

In case $d = 1$, the penalty must be larger to continuously induce the high type agent to report. There is a maximum length of the interval, as a function of the initial and final penalty of the interval, because the penalty p_t must increase at a minimum speed to ensure reporting by the high type agent.

Let $t^I(p', p)$ denote the the maximum length of the interval when the interval starts with $p_t = p$ and ends with $p_{t+t^I(p,p')} = p'$. Recall the definition of $\mathcal{P}$ in equation (3). The recursive version of the regulator's problem can now be stated as follows:

$$\mathbf{V}_\gamma(p) = \begin{cases} \sup\limits_{t,p',d} \quad \mathbf{1}_{d=0} \left(\int\limits_0^t e^{-(\gamma+r)s} \left(\int\limits_s^t e^{-(\rho+r+\lambda)q} dq \right) ds \right) + e^{-rt}\mathbf{V}_\gamma(p') \\ \text{subject to} \\ \text{if } d = 0, w^h(t) - e^{-(\rho+r)t}p' \leq -p \\ \text{if } d = 1, 0 \leq t \leq t^I(p', p) \\ p' \in \mathcal{P} \end{cases} \tag{H.1}$$

The requirement that $0 \leq t$ when $d = 1$ guarantees that $p \leq p'$. Similarly to Lemma 4, if $V^\gamma(p)$ solves this equation and the policy function $t(p)$ is such that $\inf\limits_{p \in P(\overline{p})} t(p) > 0$, then

$$V_\gamma^* = \max_{t_0, p_0} \int\limits_0^{t_0} e^{-(\gamma+r)s} \left(\int\limits_s^{t_0} e^{-(\rho+r+\lambda)q} dq \right) ds + e^{-rt_0} V_\gamma(p_0).$$

Notice that if $d = 1$, it is always optimal to set t as large as possible. Given this, and after integrating the objective, we find:

$$\mathbf{V}_\gamma(p) = \begin{cases} \sup\limits_{t,p',d} \quad \mathbf{1}_{d=0} \left(\frac{-(1-e^{-(\gamma+r)t})}{(\gamma+r)(\rho+\lambda-\gamma)} \right) + \frac{1-e^{-(\rho+\lambda+r)t}}{(\rho+\lambda-\gamma)(\rho+\lambda+r)} + e^{-rt}\mathbf{V}_\gamma(p') \\ \text{subject to} \\ \text{if } d = 0, w^h(t) - e^{-(\rho+r)t}p' \leq -p \\ \text{if } d = 1, 0 \leq t = t^I(p', p) \\ p' \in \mathcal{P} \end{cases} \tag{$R^{\gamma,*}$}$$

Computing t^I. In this step I derive the maximum length of an interval that the regulator can continuously induce reporting by high types, given a penalty p that the regulator must deliver at the beginning of the interval, i.e. the state, as well as the penalty chosen for the end of the interval, p'.

To compute the maximum length of this interval, it is sufficient to compute the path of penalties with starting point p and ending point p' such that the agent in the high state is exactly indifferent between reporting at any point on the interval. Fix some time t and suppose that the regulator wants to ensure that the agent in the high state is indifferent over all reporting times on $[t, t+s]$ for some $s > 0$.

Let τ^l be the time to transition from the high to the low state. For an agent that arrives to the model at time t_0, let τ_q be the deterministic stopping time that stops with probability 1 at the minimum of t_0+q and τ^l. Then, if the high type agent is indifferent over all stopping times on $[t, t+s]$, then it is sufficient to find a path for $p_{[t,t+s]}$ such that

$$\frac{\partial W(x^h, t, \tau_q)}{\partial q} = 0$$

for all $q \in [0, s]$. We can compute,

$$\begin{aligned}
0 &= \frac{\partial W(x^h, t, \tau_q)}{\partial q} \\
&= \frac{\partial}{\partial q}\left(\int_0^q \lambda e^{-\lambda t}\left[\frac{1-e^{-gt}}{g}(x^h - \rho\overline{p}) - e^{-gq}p_t\right]dt + e^{-\lambda t}\left(\frac{1-e^{-gq}}{g}(x^h - \rho\overline{p}) - e^{-gq}p_q\right)\right) \\
&= \frac{\partial}{\partial q}\left(\frac{x^h - \rho\overline{p}}{f}(1-e^{-fq}) - \lambda\int_0^q e^{-ft}p_t - e^{-fq}p_q\right) \\
&= (x^h - \rho\overline{p})e^{-fq} - \lambda e^{-fq}p_q + fe^{-fq}p_q - e^{-fq}\frac{\partial p_q}{\partial q} \\
&= (x^h - \rho\overline{p})e^{-fq} + ge^{-fq}p_q - e^{-fq}\frac{\partial p_q}{\partial q}
\end{aligned}$$

Solving this equation with an initial condition $p_t = p$ implies that $p_{t+q} = \frac{x^h - \rho\overline{p}}{g}(e^{gq} - 1) + e^{gq}p$.[36] So, after rearranging, we find that

$$t^I(p', p) = \frac{1}{g}ln\left(\frac{p' + \frac{x^h-\rho\overline{p}}{g}}{p + \frac{x^h-\rho\overline{p}}{g}}\right).$$

Guess solution to Equation ($R^{\gamma,*}$). I guess that an optimal policy in Equation ($R^{\gamma,*}$) is

$$\begin{cases} \text{if } p < \frac{\rho\overline{p}}{\rho+r} - \frac{x^h}{\rho+r+\lambda}, & d^*(p) = 1, \quad p'^{,*} = \frac{\rho\overline{p}}{\rho+r} - \frac{x^h}{\rho+r+\lambda}, \quad t^*(p) = t^I(p'^{,*}(p), p) \\ \text{if } p = \frac{\rho\overline{p}}{\rho+r} - \frac{x^h}{\rho+r+\lambda}, & d^*(p) = 0, \quad p'^{,*}(p) = \frac{\rho\overline{p}}{\rho+r} - \frac{x^h}{\rho+r+\lambda}, \quad t^*(p) = \infty \end{cases} \tag{H.2}$$

[36] It can be verified that p_q must be differentiable. One can proceed with only the knowledge that $\lambda\int_0^q e^{-ft}p_q dt + e^{-fq}p_q$ is differentiable, which follows immediately from differentiability of $W(x^h, t, \tau_q)$ and $\frac{x^h-\rho\overline{p}}{f}(1-e^{-fq})$, and arrive at the same conclusion.

Let $V^*_\gamma(p)$ be the value function associated to this policy. The second part of the guess can be immediately verified, since $t^*(p) = \infty$ and $d^*(p) = 0$ is the only feasible policy available when $p = \frac{\rho\overline{p}}{\rho+r} - \frac{x^h}{\rho+r+\lambda}$, and the choice of $p'^{,*}(p)$ in this case does not enter the regulator's value.

So, I need to verify the first part of the guess.

Verification. To verify the first part of the guess, it is sufficient to consider only deviations with $d = 0$. Any deviation with $d = 1$ but $p' < p'^{,*}(p)$, delivers the regulator exactly the same value as the guess, because it generates exactly the same policy, since the regulator returns to the guess after the one-shot deviation.[37]

Let $f = \rho + r + \lambda$ and $g = \rho + r$. The regulator's value under the guessed policy is

$$V^*_\gamma(p) = -e^{-(\gamma+r)(t^I(p,p'^{,*}(p)))} \int_0^\infty \left(e^{-(\gamma+r)t} \int_t^\infty e^{-(\rho+\lambda+r)s} ds \right) dt$$

$$V^*_\gamma(p) = -e^{-(\gamma+r)(t^I(p,p'^{,*}(p))} \frac{1}{f(\gamma+r)}$$

Then, to verify the guess, I must show that:

$$\frac{-e^{-(\gamma+r)(t^I(p,p'^{,*}(p))}}{f(\gamma+r)} = \begin{cases} \sup\limits_{t,p'} \quad \frac{-(1-e^{-(\gamma+r)t})}{(\gamma+r)(\rho+\lambda-\gamma)} + \frac{1-e^{-(\rho+\lambda+r)t}}{(\rho+\lambda-\gamma)f} - e^{-(r+\gamma)(t+t^I(p',p'^{,*}(p'))} \frac{1}{f(\gamma+r)} \\ \text{subject to} \\ w^h(t) - e^{-(\rho+r)t}p' \leq -p \\ p' \in P(\overline{p}) \end{cases}$$

Since $t^I(p', p'^{,*}(p')$ is decreasing in p', the optimal choice of p' given t is the maximum p' that satisfies incentive constraints. This implies that, given t, we can recover p',

$$p' = (w^h(t) + p)e^{(\rho+r)t}$$

Now, let $s(t,p) = t^I\left((w^h(t) + p)e^{(\rho+r)t}, p'^{,*}((w^h(t) + p)e^{(\rho+r)t})\right)$. Plugging this into our equation, I must show

$$\frac{-e^{-(\gamma+r)s(0,p)}}{f(\gamma+r)} = \begin{cases} \sup\limits_{t,p'} \quad \frac{-(1-e^{-(\gamma+r)t})}{(\gamma+r)(\rho+\lambda-\gamma)} + \frac{1-e^{-(\rho+\lambda+r)t}}{(\rho+\lambda-\gamma)f} - \frac{1}{f(\gamma+r)}e^{-(r+\gamma)(t+s(t,p))} \\ \text{subject to} \\ p' \in \mathcal{P} \end{cases} \qquad (R_0^{\gamma,*})$$

[37] To see this, observe that for any $p' \in [p, p'']$, $t^I(p, p'') = t^I(p, p') + t^I(p', p'')$.

Denote by $v^\gamma(t)$ the objective on the right-hand side. Since the left-hand side is the objective of the right-hand side evaluated at $t = 0$, the verification will be complete if we can show that the derivative of the objective on the right-hand side is negative everywhere.

Plugging in the definition of $p^{\prime,*}$,

$$\begin{aligned} s(t,p) &= \frac{1}{g} ln \left(\frac{x^h(\frac{1}{g} - \frac{1}{f})}{e^{gt}\left(\frac{1-e^{-ft}}{f}x^h - \frac{\rho\overline{p}}{g}\left(1-e^{-gt}\right) + p\right) + \frac{x^h - \rho\overline{p}}{g}} \right) \\ &= \frac{1}{g} ln \left(\frac{x^h(\frac{1}{g} - \frac{1}{f})}{e^{gt}\left(\frac{1-e^{-ft}}{f}x^h - \frac{\rho\overline{p}}{g} + p\right) + \frac{x^h}{g}} \right) \end{aligned}$$

Then,

$$\begin{aligned} v^\gamma(t) &= \frac{-(1-e^{-(\gamma+r)t})}{(\gamma+r)(\rho+\lambda-\gamma)} + \frac{1-e^{-ft}}{(\rho+\lambda-\gamma)f} - \frac{1}{f(\gamma+r)} e^{-(r+\gamma)(t+s(t,p))} \\ &= \frac{-(1-e^{-(\gamma+r)t})}{(\gamma+r)(\rho+\lambda-\gamma)} + \frac{1-e^{-ft}}{(\rho+\lambda-\gamma)f} - \frac{e^{-(r+\gamma)t}}{f(\gamma+r)} \left(\frac{e^{gt}\left(\frac{1-e^{-ft}}{f}x^h - \frac{\rho\overline{p}}{g} + p\right) + \frac{x^h}{g}}{x^h(\frac{1}{g} - \frac{1}{f})} \right)^{\frac{\gamma+r}{\rho+r}} \\ &= \frac{-(1-e^{-(\gamma+r)t})}{(\gamma+r)(\rho+\lambda-\gamma)} + \frac{1-e^{-ft}}{(\rho+\lambda-\gamma)f} - \frac{1}{f(\gamma+r)} \left(\frac{\frac{1-e^{-ft}}{f}x^h - \frac{\rho\overline{p}}{g} + p + e^{-gt}\frac{x^h}{g}}{x^h(\frac{1}{g} - \frac{1}{f})} \right)^{\frac{\gamma+r}{\rho+r}} \end{aligned}$$

The last term in parentheses in the second line is always positive for feasible p', i.e. $p' \in P(\overline{p})$, and is smaller than 1. In the last line then, the last term in parentheses is always positive for feasible p' and is smaller than $e^{-(\rho+r)t}$. Differentiating $v^\gamma(t)$, we find:

$$\begin{aligned} \frac{\partial v^\gamma(t)}{\partial t} &= \frac{1}{(\rho+\lambda-\gamma)}\left(e^{-ft} - e^{-(\gamma+r)t}\right) - \frac{e^{-ft} - e^{-gt}}{\lambda} \left(\frac{p + \frac{x^h e^{-gt}}{g} + \frac{1-e^{-ft}}{f}x^h - \frac{\rho\overline{p}}{g}}{\frac{x^h\gamma}{fg}} \right)^{\frac{\gamma-\rho}{\rho+r}} \\ &\leq \frac{1}{\rho+\lambda-\gamma}\left(e^{-ft} - e^{-(\gamma+r)t}\right) - \frac{1}{\lambda}\left(e^{-ft} - e^{-gt}\right) e^{-(\gamma-\rho)t} \end{aligned}$$

where the inequality follows since, as noted above, the last term in parentheses in the first line is positive and strictly smaller than e^{-gt}. Rearranging, we find

$$\begin{aligned} \frac{\partial v^\gamma(t)}{\partial t} &\leq \frac{1}{\rho+\lambda-\gamma}\left(e^{-ft} - e^{-(\gamma+r)t}\right) - \frac{1}{\lambda}\left(e^{-ft} - e^{-gt}\right) e^{-(\gamma-\rho)t} \\ &= e^{-(\gamma+r)t}\left(\frac{1}{\lambda} - \frac{1}{\rho+\lambda-\gamma}\right) + e^{-ft}\left(\frac{1}{\rho+\lambda-\gamma} - \frac{e^{-(\gamma-\rho)t}}{\lambda}\right) \\ &= e^{-(\gamma+r)t}\left(\frac{1}{\lambda} - \frac{1}{\rho+\lambda-\gamma}\right) + e^{-ft}\left(\frac{1}{\rho+\lambda-\gamma} - \frac{1}{\lambda}\right) \end{aligned}$$

$$= \frac{e^{-ft}}{\lambda}\left[\overbrace{\frac{\lambda(1-e^{-(\gamma-\rho)t})+(\rho-\lambda)(e^{(\rho+\lambda-\gamma)t}-e^{(\rho-\gamma)t})}{\rho+\lambda-\gamma}}^{c(t)\equiv}\right]$$

The term $c(t)$ is 0 at $t=0$ and we can differentiate to find,

$$\begin{aligned}
\frac{\partial c(t)}{\partial t} &= \frac{\lambda(\gamma-\rho)e^{-(\gamma-\rho)t}+(\rho-\gamma)^2e^{-(\gamma-\rho)t}(e^{\lambda t}-1)+\lambda(\rho-\gamma)e^{(\lambda+\rho-\gamma)t}}{\rho+\lambda-\gamma} \\
&= \frac{(\gamma-\rho)e^{-(\gamma-\rho)t}\left[\lambda+(\gamma-\rho)[e^{\lambda t}-1]-e^{\lambda t}\lambda\right]}{\rho+\lambda-\gamma} \\
&= (\gamma-\rho)e^{-(\gamma-\rho)t}(1-e^{\lambda t}) \\
&\leq 0
\end{aligned}$$

where the last line is a result of the fact that $\gamma > \rho$ and $1-e^{\lambda t} \leq 0$. This implies then that

$$\frac{\partial v^{\gamma}(t)}{\partial t} \leq 0$$

which subsequently implies that $V_{\gamma}^*(p)$ solves Equation ($R_0^{\gamma,*}$). This completes the verification and so $V_{\gamma}^*(p)$ solves Equation ($R^{\gamma,*}$).

Conclusion. Given our solution to the recursive representation $V_{\gamma}^*(p)$, and observing that $V_{\gamma}^*(p)$ is maximized at $p=\underline{p}$, we know that

$$V_{\gamma}^* = \max_{t_0}\int_0^{t_0} e^{-(\gamma+r)s}\left(\int_s^{t_0} e^{-(\rho+r+\lambda)q}dq\right)ds + e^{-rt_0}V_{\gamma}^*(\underline{p}).$$

Applying the policy functions associated with V_{γ}^* that were verified above then generates the optimal path described in the theorem. $\square$