Online Appendix
Dynamic Amnesty Programs

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E Other Proofs of Results in Appendix C

Proof of Lemma C.1: Observe that for $s \in (\underline{t}, t]$, μ^h_s faces an outflow rate of $(\rho+\lambda)\mu^h_s$ and an inflow rate of 1, where ρ is the risk of detection and λ the risk of transition to the low state. That is

$$\frac{\partial \mu^h_s}{\partial s} = 1 - (\rho+\lambda)\mu^h_s.$$

Solving this differential equation with the initial condition $\mu^h_0 = 0$ leads to the result. $\square$

Proof of Lemma C.2: For any policy $(\mathbf{p}, \mathbf{a}) \in \mathcal{M}$, define $\tilde{p}_t \equiv \left(\mathbb{1}_{a_t(x^h)=0}\right)\overline{p} + (1 - \mathbb{1}_{a_t(x^h)=0})p_t$ and $\tilde{a}_t(x) \equiv a_t(x^h)$ for $x \in \{x^h, x^l\}$. The resulting policy, $(\tilde{\mathbf{p}}, \tilde{\mathbf{a}})$, satisfies the constraints on the right-hand side of (C.6) and delivers the regulator the same value as $(\mathbf{p}, \mathbf{a})$ when $\alpha_l = 0$. On the other hand, any policy that satisfies the constraints on the right-hand side of (C.6) is an element of $\mathcal{M}$, and the result follows. $\square$

Proof of Lemma C.3: For any policy $(\mathbf{p}, \mathbf{a}) \in \mathcal{M}$, let $\mathcal{T}(\mathbf{a}) \equiv \{t | a_t(x^h) = 1\}$. Let $\mathcal{M}^0 \subset \mathcal{M}$ be the set of policies $(\mathbf{p}, \mathbf{a})$ such that

(i) $\left(1 - a_t(x^h)\right)\overline{p} = \left(1 - a_t(x^h)\right)p_t$ and

(ii) $\inf\limits_{\substack{(t,s)\in(\mathcal{T}(\mathbf{a}))^2 \\ \text{s.t. } t\neq s}} |t-s| > 0.$

Let $\mathbf{t}(\mathbf{a}) \equiv (t_i(\mathbf{a}))_{i\in\mathbb{N}}$ be the increasing sequence such that $\bigcup\limits_{i\in\mathbb{N}} t_i(\mathbf{a}) = \mathcal{T}(a)$. I first show that

$$V^* = \sup_{(\mathbf{p},\mathbf{a})\in\mathcal{M}^0} V(\mathbf{p}, \mathbf{a}). \tag{E.1}$$

To see this, fix any policy $(\mathbf{p}, \mathbf{a}) \in \mathcal{M} \cap (\mathcal{M}^0)^C$. Choose recursively a sequence,

- $\tilde{t}_0 \in \Big[\inf \mathcal{T}(\mathbf{a}), \epsilon + \inf \mathcal{T}(\mathbf{a})\Big] \cap \mathcal{T}(\mathbf{a})$
- $\tilde{t}_{i+1} \in \Big[\inf\left(\mathcal{T}(\mathbf{a}) \cap \left[\tilde{t}_i + \epsilon, \infty\right)\right), \epsilon + \inf\left(\mathcal{T}(\mathbf{a}) \cap \left[\tilde{t}_i + \epsilon, \infty\right)\right)\Big] \cap \mathcal{T}(\mathbf{a})$.

and generate policy $(\tilde{\mathbf{p}}, \tilde{\mathbf{a}})$ by setting $\tilde{p}_t \equiv \overline{p} + (p_t - \overline{p})\mathbb{1}_{t\in\{\tilde{t}_i\}_{i\in\mathbb{N}}}$ and $\tilde{a}_t(x) = \mathbb{1}_{t\in\{\tilde{t}_i\}_{i\in\mathbb{N}}}$ for each $x \in \{x^h, x^l\}$. Observe that $(\tilde{\mathbf{p}}, \tilde{\mathbf{a}}) \in \mathcal{M}^0$. Since regulator discounts at rate $r > 0$ and $\alpha_l = 0$, $|V(\mathbf{p}, \mathbf{a}) - V(\tilde{\mathbf{p}}, \tilde{\mathbf{a}})| \to_\epsilon 0$ and (E.1) follows.

The remainder of the proof is a dynamic programming principle, which I present for completeness. I will show that if $\alpha_l = 0$ and $\mathbf{V}(p)$ satisfies the premise of the lemma with associated policies $(t_V(p), p'_V(p))$ then,

$$\sup_{(\mathbf{p},\mathbf{a})\in\mathcal{M}^0} V(\mathbf{p},\mathbf{a}) = \max_{\substack{t_0\geq 0,\\ p_0\in\mathcal{P}}} \left\{-v(t_0) + e^{-rt_0}\mathbf{V}(p_0)\right\}. \tag{E.2}$$

The result then follows from (E.1). To prove (E.2) holds, I first show

$$\sup_{(\mathbf{p},\mathbf{a})\in\mathcal{M}^0} V(\mathbf{p},\mathbf{a}) \leq \max_{\substack{t_0\geq 0,\\ p_0\in\mathcal{P}}} \left\{-v(t_0) + e^{-rt_0}\mathbf{V}(p_0)\right\}. \tag{E.3}$$

For any policy $(\mathbf{p},\mathbf{a})$, letting $\delta_i(\mathbf{a}) \equiv t_i(\mathbf{a}) - t_{i-1}(\mathbf{a})$,

$$\begin{aligned} -V(\mathbf{p},\mathbf{a}) &= \sum_{i=0}^{\infty} e^{-rt_{i-1}(\mathbf{a})} \int_0^{\delta_i(\mathbf{a})} e^{-rt}\mu^h_{t+t_{i-1}(\mathbf{a})}dt = \sum_{i=0}^{\infty} e^{-rt_{i-1}(\mathbf{a})} \int_0^{\delta_i(\mathbf{a})} e^{-rt}\frac{1-e^{-(\rho+\lambda)t}}{\rho+\lambda}dt \\ &= \sum_{i=0}^{\infty} e^{-rt_{i-1}} v\left(\delta_i(\mathbf{a})\right) \end{aligned}$$

where $t_{-1}(\mathbf{a}) = 0$, the second equality follows from Lemma C.1 and the third equality follows from equation (C.3). Inequality (E.3) follows by observing that $(\mathbf{p},\mathbf{a}) \in \mathcal{M}^0$ if and only if for each $i \in \mathbb{N}$, $w^h(t_i) - e^{-(\rho+r)\delta_{i+1}(\mathbf{a})}p_{t_{i+1}(\mathbf{a})} \leq -p_{t_i(\mathbf{a})}$.

Next, I argue that

$$\sup_{(\mathbf{p},\mathbf{a})\in\mathcal{M}^0} V(\mathbf{p},\mathbf{a}) \geq \max_{\substack{t_0\geq 0,\\ p_0\in\mathcal{P}}} \left\{-v(t_0) + e^{-rt_0}\mathbf{V}(p_0)\right\}. \tag{E.4}$$

For any $p_0 \in \mathcal{P}$, recursively define $p^i(p_0) \equiv p'_V(p^{i-1}(p_0))$ and $p^0(p_0) = p_0$. For any $p_0 \in \mathcal{P}$ and $t_0 \geq 0$, recursively define $t^i(p_0) = t^{i-1}(p_0) + t_V(p^{i-1}(p_0))$ and $t^0(p_0) = t_0$. Then, for any choice p_0 and t_0 on the right-hand side of (E.4), define $(\mathbf{p},\mathbf{a})$ as follows,

$$p_t = \overline{p} + \sum_{i\in\mathbb{N}} \mathbb{1}_{t=t^i(p_0)}\left(p^i(p_0) - \overline{p}\right) \tag{E.5}$$

$$a_t(x) = \sum_{i\in\mathbb{N}} \mathbb{1}_{t=t^i(p_0)} \tag{E.6}$$

Then since $\inf_{i\in\mathbb{N}} (t^i(p_0) - t^{i-1}(p_0)) > 0$, $(\mathbf{p},\mathbf{a}) \in \mathcal{M}^0$. Further, repeatedly substituting yields $V(\mathbf{p},\mathbf{a}) = -v(t_0) + e^{-rt_0}\mathbf{V}(p_0)$. As a result, (E.4) is satisfied. Combining with (E.3) and observing that $\mathbf{V}(p_0)$ is maximized at $\underline{p}$ (since all policies feasible for some $p \in \mathcal{P}$ are feasible for $\underline{p}$) completes the proof. $\square$

Proof of Lemma C.4: First, observe

$$\lim_{t\to\infty}\left(w^h(t)-e^{-(\rho+r)t}\underline{p}\right)=\frac{x^h-x^l}{\rho+r+\lambda}-\frac{\rho\overline{p}-x^l}{\rho+r}=\frac{x^h-x^l}{\rho+r+\lambda}-\Delta_l-\underline{p}\leq-\underline{p} \tag{E.7}$$

where the inequality follows since $\theta\in\Theta^*$, and is strict whenever $\theta\in(\Theta^*)^o$. Further,

$$w^h(0)-e^{-(\rho+r)0}\underline{p}=-\underline{p}. \tag{E.8}$$

Observe next that

$$\frac{\partial}{\partial t}\left(w^h(t)-e^{-(\rho+r)t}\underline{p}\right)=e^{-(\rho+r)t}\left((x^h-x^l)e^{-\lambda t}+(\rho+r)\underline{p}-(\rho\overline{p}-x^l)\right) \tag{E.9}$$

Since $\theta\in\Theta^*$, equation (E.9) is strictly positive at 0 and crosses zero exactly once. Combining (E.8), (E.9) and the strict version of (E.7) implies that for $\theta\in(\Theta^*)^o$, equation (C.8) has a unique strictly positive solution $t(p)>0$, which is strictly increasing. If instead $\theta\in\partial\Theta^*$, then $\mathcal{P}=\{-\underline{p}\}$ and equation (C.8) has a unique non-zero solution at $t(p)=\infty$. Since $t(p)$ is strictly increasing, $\inf_{p\in\mathcal{P}} t(p)=t(\underline{p})>0$.

Finally, for any $p\in\mathcal{P}$ and $t\geq t(p)$, (E.9) is non-positive, so $w^h(t)-e^{-(\rho+r)t}\underline{p}\leq -p$. □

Proof of Lemma C.5: Let $f=\rho+r+\lambda$. Plugging in definitions yields,

$$v^0(t)+e^{-rt}\frac{v^0(\underline{t})}{1-e^{-r\underline{t}}}=\frac{1}{\rho+\lambda}\left(\frac{1-e^{-ft}}{f}+e^{-rt}\frac{1-e^{-f\underline{t}}}{f(1-e^{-r\underline{t}})}\right)$$

Differentiate the term in parentheses on the right-hand side with respect to t to get:

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{1-e^{-ft}}{f}+\frac{1-e^{-f\underline{t}}}{f(1-e^{-r\underline{t}})}\right)&=e^{-ft}-\frac{r}{f}e^{-rt}\frac{1-e^{-f\underline{t}}}{1-e^{-r\underline{t}}}\\&\leq e^{-ft}-\frac{r}{f}e^{-rt}\frac{1-e^{-ft}}{1-e^{-rt}}\\&=\frac{1-e^{-ft}}{f}\left[\frac{fe^{-ft}}{1-e^{-ft}}-\frac{re^{-rt}}{1-e^{-rt}}\right]\\&=\frac{1-e^{-ft}}{f}\left[\phi(f)-\phi(r)\right]\end{aligned}$$

where $\phi(a)\equiv\frac{ae^{-at}}{1-e^{-at}}$ and the second line follows for any $t\geq\underline{t}$ because $\frac{1-e^{-ft}}{1-e^{-rt}}$ is decreasing in

t.[53] To see that $\phi(f) - \phi(r) \leq 0$, note that $f > r$ and

$$\frac{\partial}{\partial a}\phi(a) = \frac{e^{at}\left(1 - (at + e^{-at})\right)}{(e^{at}-1)^2} \\ \leq 0$$

for any $at \geq 0$ since $z + e^{-z} > 1$ for any $z \geq 0$. This concludes the first part of the lemma.

For the second part of the lemma, observe that, by Lemma C.4, $t(p)$ defined by equation (C.8) is increasing in p. Applying the first part of this lemma then leads to the result. □

F Randomized Policies

I have restricted the regulator to deterministic policies. Although I do not characterize the optimal policy for general random policies, I expand the model to allow for a limited class of random policies and show that the deterministic optimal policy remains optimal. Extend V linearly to random policies.

Definition 2. *A randomized policy $(\boldsymbol{p}, \boldsymbol{a})$ is called a γ-Poisson policy if there exists t_0 and a sequence of random variables $(t_i)_{i\in\mathbb{N}}$ s.t.*

- *$t_{i+1} - t_i$ is independent of t_i and exponentially distributed with rate parameter γ*
- *$p_{t_i} = \underline{p}$ for $i \in \mathbb{N}$ and $p_t = \overline{p}$ otherwise*
- *$a_t(x^h) = 1$ if and only if $t \in \{t_i\}_{i\in\mathbb{N}}$*

The set of γ-Poisson policies for any $\gamma > 0$ is denoted Γ.

These policies feature inter-arrival times of minimum penalties that are exponentially distributed with mean $\frac{1}{\gamma}$. I restrict to the setting in which $\underline{p} = x^l = \alpha_l = 0$ and argue that the policy in Theorem 1 remains optimal when allowing the regulator to choose from Γ. Let $\mathcal{M}^\Gamma \equiv \mathcal{M} \cup \Gamma$.

$$(\mathcal{P}^\Gamma) \qquad V^\Gamma \equiv \sup_{(\mathbf{p},\mathbf{a})\in\mathcal{M}^\Gamma} V(\mathbf{p},\mathbf{a})$$

i.e. the expanded regulator's problem allowing for policies in Γ (with some abuse of notation since $\mathbf{p}$ and $\mathbf{a}$ are now random variables).

[53] To see this, let $z(t) \equiv e^{-ft}$ and $a \equiv \frac{r}{f}$, so that $\frac{1-e^{-ft}}{1-e^{-rt}} = \frac{1-z(t)}{1-z(t)^a}$. Let $\zeta(z) \equiv \frac{1-z}{1-z^a}$. Then $\frac{\partial \frac{1-e^{-ft}}{1-e^{-rt}}}{\partial t} = \zeta'(z(t))(-fz(t))$. Compute $\zeta'(z)$ to get $\zeta'(z) = \frac{-(1-z^a)+a(z^{a-1}-z^a)}{(1-z^a)^2}$; the numerator is decreasing in z, so to show that $\zeta'(z) \geq 0$ (and hence $\frac{\partial \frac{1-e^{-ft}}{1-e^{-rt}}}{\partial t} \leq 0$), it is sufficient to show that $\zeta'(1) \geq 0$, which follows by applying L'hôspital's rule twice. Combining this with the chain rule above leads to the conclusion.

I prove the result below for the case of $\alpha_l = x^l = \underline{p} = 0$, but it extends readily to the general case.

Theorem F.1. *Suppose $\underline{p} = x^l = \alpha_l = 0$. Then*

$$V^* = V^\Gamma > V(\boldsymbol{p}, \boldsymbol{a})$$

for any $(\boldsymbol{p}, \boldsymbol{a}) \in \Gamma$, where V^* is the regulator's optimal value over deterministic policies.

Proof. Let $f \equiv \rho + r + \lambda$. In a γ-Poisson policy, the recommendation $a_t(x^h) = \mathbb{1}_{p_t=0}$ is incentive compatible if and only if

$$\mathbb{E}_\gamma\left(w^h(t)\right) \leq 0 \tag{F.1}$$

where $\mathbb{E}_\gamma$ denotes the expectation operator for t distributed as an exponential distribution with rate parameter γ. Recalling the relationship between $\mathbf{V}$ and v (defined in equation (C.3)) from Lemma C.3, the result follows if any γ-Poisson policy satisfying equation (F.1) also satisfies

$$\sup_{t_0 \geq 0} -v(t_0) + e^{-rt_0}\frac{-\mathbb{E}_\gamma\left(v(t)\right)}{\mathbb{E}_\gamma\left(1-e^{-rt}\right)} < \sup_{t_0 \geq 0} -v(t_0) + e^{-rt_0}\frac{-v(t(0))}{1-e^{-rt(0)}}$$

where $t(0)$ is the unique strictly positive solution to equation (C.8) at $p = \underline{p} = 0$ (when $\theta \in \Theta^*$). Since the choice of t_0 has the same domain in both problems, and a solution $t_0 \in \mathbb{R}_+$ exists for both problems, it is sufficient to show that

$$-\frac{\mathbb{E}_\gamma\left(v(t)\right)}{\mathbb{E}_\gamma\left(1-e^{-rt}\right)} < -\frac{v(t(0))}{1-e^{-rt(0)}} \tag{F.2}$$

From the definition of $t(0)$, $\frac{x^h v^0(t(0))}{1-e^{-(\rho+r)t(0)}} = \frac{\rho\overline{p}}{\rho+r}$, where $v^0(t) = \frac{1-e^{-ft}}{f(\rho+\lambda)}$. Recall that $w^h(t) = x^h(\rho+\lambda)v^0(t) - \frac{\rho\overline{p}}{\rho+r}(1-e^{-(\rho+r)t})$. Then, inequality (F.1) becomes

$$\begin{aligned}
&\mathbb{E}_\gamma\left(w^h(t)\right) \leq 0\\
\iff &\mathbb{E}_\gamma\left(x^h\frac{1-e^{-ft}}{f} - \frac{\rho\overline{p}}{\rho+r}(1-e^{-(\rho+r)t})\right) \leq 0\\
\iff &\mathbb{E}_\gamma\left(\left(1-e^{-ft}\right) + e^{-(\rho+r)t}\frac{\left(1-e^{-ft(0)}\right)}{1-e^{-(\rho+r)t(0)}}\right) \leq \frac{1-e^{-ft(0)}}{1-e^{-rt(0)}}
\end{aligned} \tag{F.3}$$

For inequality (F.2), plugging in (C.9) ($v(t) = \frac{1-e^{-rt}}{r(\rho+\lambda)} - v^0(t)$) and rearranging yields

$$-\frac{\mathbb{E}_\gamma\left(v(t)\right)}{\mathbb{E}_\gamma\left(1-e^{-rt}\right)} < -\frac{v(t(0))}{1-e^{-rt(0)}}$$

$$\text{(F.4)} \qquad \iff \mathbb{E}_\gamma\left(1 - e^{-ft} + e^{-rt}\frac{1-e^{-ft(0)}}{1-e^{-rt(0)}}\right) < \frac{1-e^{-ft(0)}}{1-e^{-rt(0)}}$$

Now, observe that inequalities (F.4) and (F.3) are special cases of the inequality:

$$\text{(F.5)} \qquad \underbrace{\mathbb{E}_\gamma\left((1-e^{-ft})(1-(z)^a) + (e^{-ft})^a(1-z)\right) - (1-z)}_{h(a,z,\gamma)\equiv} \leq 0$$

where $a = \frac{r}{\rho+r+\lambda}$ for the regulator and $a = \frac{\rho+r}{\rho+r+\lambda}$ for the agent, and $z \equiv e^{-ft(0)}$ (and $\leq$ is replaced with $<$ for inequality F.4). Similar to Proposition C.1, the crucial step is:

(C^γ) if $h(a,z,\gamma) \leq 0$ at some $\overline{a} \in (0,1)$, then $h(a,z,\gamma) < 0$ for each $0 < a \leq \overline{a}$.

With this the proof will be concluded, since any policy that satisfies (F.1) also satisfies (F.2). Integrate $h(a,z,\gamma)$ with respect to t,

$$(1-z^a)\left(1 - \frac{\gamma}{\gamma+f}\right) + \frac{\gamma}{\gamma+fa}(1-z) - (1-z) \leq 0.$$

Rather than show (C^γ) for h, I will show it for $\tilde{h}(a,z,\gamma) = h(a,z,\gamma)\times(\gamma+fa)$, from which the property for h can be recovered (since for $a \in (0,1)$, $sgn(h) = sgn(\tilde{h})$). Computing $\tilde{h}$,

$$\tilde{h} = (1-z^a)\frac{(\gamma+fa)f}{\gamma+f} + \gamma(1-z) - (1-z)(\gamma+fa)$$

I claim that if $\frac{\partial^2\tilde{h}}{\partial a^2}$ has at most one 0, then (C^γ) will be verified and the proof will be complete. To see this, observe that as $a \downarrow -\infty$, $\tilde{h} \uparrow \infty$. Observe also that $\tilde{h}(0) = \tilde{h}(1) = 0$. To violate the property, there must exist points $0 < a_1 < a_2 < 1$ such that $\tilde{h}(a_1) \geq 0$, $\tilde{h}(a_2) \leq 0$, while $\tilde{h}(0) = \tilde{h}(1) = 0$. Since $\tilde{h} \uparrow \infty$ as $a \downarrow -\infty$, there must also exist $a_0 < 0$ s.t. $h(a_0) > 0$. Satisfying all of these requires $\frac{\partial^2\tilde{h}}{\partial\tilde{h}^2}$ to have *at least two* zeros. So, I proceed to show that $\frac{\partial^2\tilde{h}}{\partial a^2}$ has at most one 0.

Twice differentiating $\tilde{h}$ leads to:

$$\frac{\partial^2\tilde{h}}{\partial a^2} = -\left(\frac{f}{\gamma+f}\right) z^a ln(z)\left[2f + ln(z)(\gamma+fa)\right]$$

which has at most one 0. So, I conclude that $\frac{\partial^2\tilde{h}}{\partial a^2}$ crosses 0 at most one zero so (C^γ) holds, and the conclusion follows. □

G Generalizing the Arrival Distribution

In this section, I assume that the regulator faces a stream of agents arriving at time-inhomogeneous rate $e^{-\gamma t}$ for some $\gamma \in [0, \infty)$. The model studied in Section II corresponds to $\gamma = 0$, while $\gamma > 0$ corresponds to a setting in which the distribution of arrival is weighted towards time 0. I show that when $\gamma < \rho$, the main theorem of Section II still holds; an optimal policy consists of amnesty cycles that take the form described in Theorem 1. When instead $\gamma > \rho$, a new optimal policy can be described as follows: after an initialization period as in Theorem 1, the regulator offers an interval with an *increasing* self-reporting penalty, and after this interval offers a fixed penalty forever.

I operate in this section under the assumption that $\underline{p} = x^l = 0$, but this is only for simplicity and all of the results generalize.

Let $V_\gamma(\mathbf{p}, \mathbf{a})$ denote the regulator's value from a policy $(\mathbf{p}, \mathbf{a})$ when the arrival rate of agents is $e^{-\gamma t}$ for $\gamma \in [0, \infty)$. Then, as in Section I, the regulator solves

$$V_\gamma^* \equiv \sup_{(\mathbf{p},\mathbf{a}) \in \mathcal{M}} V_\gamma(\mathbf{p}, \mathbf{a}).$$

The steps for proving Theorem 1 apply with little adjustment to V_γ^*, as long as $\gamma \leq \rho$.

Proposition G.1. *Suppose $\gamma \leq \rho$ and $\underline{p} = x^l = 0$. Then, the policy in Theorem 1 remains optimal.*

When $\gamma \leq \rho$, the arrival rate of agents is still relatively steady over time, and the fact that agents arrive more quickly near time 0 is not enough to overcome the backloading motive that leads to the cyclical optimal policy. The proof is given below.

This is no longer true when $\gamma > \rho$. In this case, the arrival of agents is front-loaded and the policy described in Theorem 1 does not deliver the regulator's optimal value. After the choice of the first reporting time, the optimal policy takes the following form:

(*i*) an interval with an *increasing* self-reporting penalty, on which all types report,

(*ii*) an upward jump at the end of this interval and

(*iii*) afterwards, a constant self-reporting penalty, with only low types reporting.

The proposition below states the form of the optimal policy. When $\theta \notin \Theta^*$, a static policy is again optimal, so I restrict the proposition to the case $\theta \in \Theta^*$. Let

$$t^I \equiv ln \left(\frac{x^h - \frac{(\rho+r)x^h}{\rho+r+\lambda}}{x^h - \rho\overline{p}} \right) \frac{1}{\rho + r}.$$

Proposition G.2. *Suppose $\gamma > \rho$, $\underline{p} = x^l = 0$, and $\theta \in \Theta^*$. Then, there exists t_0 such that an optimal policy, $(\boldsymbol{p}, \boldsymbol{a}) = ((p_t^*), (a_t^*))_{t \geq 0}$, is:*

- $p_t^* = (1 - e^{-(\rho+r)(t_0-t)})\frac{(\rho\overline{p})}{\rho+r}$ *for $t < t_0$*
- $p_t^* = (e^{(\rho+r)(t-t_0)} - 1)\frac{x^h - \rho\overline{p}}{\rho+r}$ if $t_0 \leq t \leq t_0 + t^I$ *and*
- $p_t^* = \frac{\rho\overline{p}}{\rho+r}$ *for $t \geq t_0 + t^I$.*
- $a_t^*(x^h) = 1$ *if and only if $t_0 \leq t \leq t_0 + t^I$*
- $a_t^*(x^l) = 1$ *for all t*

The result is proved below. An example of the optimal policy in Proposition G.2 beyond t_0 is depicted in Figure 2. As in Theorem 1, the existence of t_0 is a result of the fact that the regulator has no prior incentive constraints to satisfy until the initial amnesty offer.

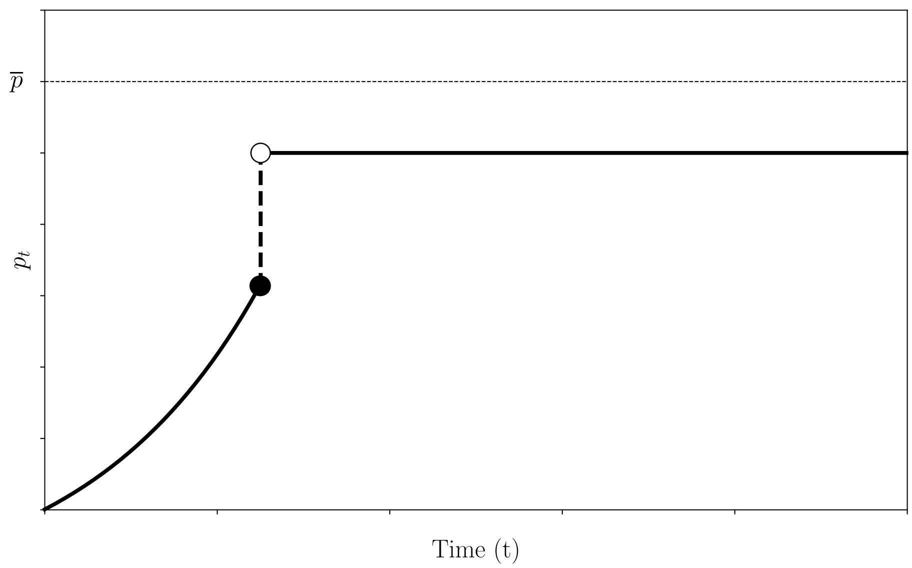

Figure 2: An Example of the Optimal Policy in Proposition G.2

Proof of Proposition G.1: Suppose $\gamma < \rho$. Lemma C.2 proceeds in exactly the same way. The statement of Lemma C.3 must now be altered so that rather than applying a discount of e^{-rt} the regulator applies a discount of $e^{-(\gamma+r)t}$, but is otherwise identical. To see this, fix some $t \geq 0$ and policy $(\mathbf{p}, \mathbf{a}) \in \mathcal{M}^0$. Recall that, by the definition of $\mathcal{M}^0$, there exists a sequence $\mathbf{t} = (t_i)_{i \in \mathbb{N}}$ such that $a_t(x^h) = 1$ if and only if $t \in \{t_i\}_{i \in \mathbb{N}}$. , Then note that

μ_t^h for $t \in (t_i, t_{i+1})$ is now

$$\mu_t^h = \int_0^{t-t_i} e^{-\gamma s} e^{-(\rho+\lambda)(t-t_i-s)} ds$$
$$= \frac{e^{-\gamma(t-t_i)} - e^{-(\rho+\lambda)(t-t_i)}}{\rho+\lambda-\gamma}$$

Plugging in to compute the regulator's value yields,

$$V_\gamma(\mathbf{p}, \mathbf{a}) = -\sum_{i=0}^{\infty} \int_{t_i}^{t_{i+1}} e^{-rt} \frac{e^{-\gamma(t-t_i)} - e^{-(\rho+\lambda)(t-t_i)}}{\rho+\lambda-\gamma}$$
$$= -\frac{1}{(\gamma+r)(\rho+\lambda-\gamma)} + \frac{1}{\rho+\lambda-\gamma} \sum_{i=0}^{\infty} e^{-(r+\gamma)t_{i-1}} \frac{1-e^{-(\rho+r+\lambda)(t_i-t_{i-1})}}{\rho+r+\lambda}$$

where $t_{-1} = 0$. Letting $\hat{v}(t) = \frac{1-e^{-(\rho+r+\lambda)t}}{(\rho+r+\lambda)(\rho+\lambda-\gamma)}$, a version of Lemma C.3 holds using the recursive equation

$$\mathbf{V}(p) = \begin{cases} \sup_{t \geq 0, p' \in \mathcal{P}} & \hat{v}(t) + e^{-(r+\gamma)t}\mathbf{V}(p') \\ \text{subject to} & \\ w^h(t) - e^{-(\rho+r)t}p' \leq -p & \end{cases}$$

and replacing the equation for V^* with $V_\gamma^* + \frac{1}{(\gamma+r)(\rho+\lambda-\gamma)} = \max_{t \geq 0, p_0 \in \mathcal{P}} \{\hat{v}(t) + e^{-(r+\gamma)t_0}\mathbf{V}(p_0)\}$. As long as $\gamma < \rho$, Proposition G.1 can be derived with the same steps as Theorem 1 and the result follows. □

Proof of Proposition G.2: To avoid non-generic cases, suppose that $\rho + \lambda \neq \gamma$. The result for the case $\rho + \lambda = \gamma$ can be recovered from the proof for the case $\rho + \lambda \neq \gamma$ by taking the limit and using the continuity of the regulator's value in γ.

Let V^{cont} be the regulator's value associated to the policy described in the proposition. So to prove the result I must show that

$$V^{cont} = \sup_{(\mathbf{p}, \mathbf{a}) \in \mathcal{M}^{cont}} V_\gamma(\mathbf{p}, \mathbf{a}).$$

Recursive Representation. The recursive problem is described as follows:

- A *decision node* of the regulator is any reporting time of the high return agent *that is not an interior point of an interval* of reporting times
- The *choice* of the regulator is now either

- *next reporting time* and *penalty at next reporting time* or
- *length of an interval* (I) on which to continuously induce reporting by high types ($a_t(x) = 1$ for $t \in I$) as well as the penalty offered at the end of this interval

Let $d = 0$ indicate that the regulator is choosing the former and $d = 1$ the latter.

- The *state* of the regulator is the reporting penalty that she must offer immediately
- The *constraint* of the regulator is
  - if $d = 0$, the one-shot incentive compatibility condition from the time-homogeneous case
  - if $d = 1$,
    - penalty at the end of the interval > penalty at the start of the interval
    - a bound on the maximum length of the interval

In case $d = 1$, there is a maximum length of the interval, as a function of the initial and final penalty of the interval, because the penalty p_t must increase at a minimum speed to ensure reporting by the high return agent at each instant.

Let $t^I(p, p')$ denote the maximum length of the interval when the interval starts with $p_t = p$ and ends with $p_{t+t^I(p,p')} = p'$. Recall the definition of $\mathcal{P}$ in equation (C.5). Then a solution to the recursive problem is a function $\mathbf{V}_\gamma(p)$ such that

$$\text{(G.1)} \qquad \mathbf{V}_\gamma(p) = \begin{cases} \sup\limits_{t,p',d} \quad -\mathbb{1}_{d=0} \left(\int\limits_0^t e^{-(\gamma+r)s} \left(\int\limits_s^t e^{-(\rho+r+\lambda)(q-s)} dq \right) ds \right) + e^{-rt}\mathbf{V}_\gamma(p') \\ \text{subject to} \\ \text{if } d = 0, w^h(t) - e^{-(\rho+r)t}p' \leq -p \\ \text{if } d = 1, 0 \leq t \leq t^I(p, p') \\ p' \in \mathcal{P} \end{cases}$$

with associated policy functions $d(p)$, $t(p)$ and $p'(p)$, where $t(p)$ is the length of the interval if $d = 1$ and the delay until the next reporting time if $d = 0$. Similarly to Lemma C.3, if $\mathbf{V}_\gamma(p)$ solves this equation and the policy function $t(p)$ is such that, whenever $d(p) = 0$, $\inf\limits_{p \in \mathcal{P}} t(p) > 0$, then

$$V^*_\gamma = \max_{t_0,p_0} \int\limits_0^{t_0} e^{-(\gamma+r)s} \left(- \int\limits_s^{t_0} e^{-(\rho+r+\lambda)(q-s)} dq \right) ds + e^{-rt_0}\mathbf{V}_\gamma(p_0).$$

Notice that if $d = 1$, it is always optimal to set $t = t^I(p', p)$ i.e. at the upper bound. Equation (G.1) then becomes

$$
\text{(G.2)} \qquad \mathbf{V}_\gamma(p) = \begin{cases} \sup\limits_{t,p',d} \quad \mathbb{1}_{d=0}\left(\frac{-(1-e^{-(\gamma+r)t})}{(\gamma+r)(\rho+\lambda-\gamma)} + \frac{1-e^{-(\rho+\lambda+r)t}}{(\rho+\lambda-\gamma)(\rho+\lambda+r)}\right) + e^{-rt}\mathbf{V}_\gamma(p') \\ \text{subject to} \\ \text{if } d = 0, w^h(t) - e^{-(\rho+r)t}p' \leq -p \\ \text{if } d = 1, 0 \leq t = t^I(p, p') \\ p' \in \mathcal{P} \end{cases}
$$

Computing t^I. In this step I derive the maximum length of an interval that the regulator can continuously induce reporting by high types, given a penalty p that the regulator must deliver at the beginning of the interval, i.e. the state, as well as the penalty chosen for the end of the interval, p'.

To compute the maximum length of this interval, it is sufficient to compute the path of penalties with starting point p and ending point p' such that the agent in the high state is exactly indifferent between reporting at any point on the interval. Fix some time t_0 and suppose that the regulator wants to ensure that the agent in the high state is indifferent over all reporting times on $[t_0, t_0 + s]$ for some $s > 0$.

Let τ^l be the transition time from the high to the low state. For an agent that arrives to the model at time t_0, let τ_q be the deterministic stopping time that stops with probability 1 at the minimum of $t_0 + q$ and τ^l. Then, to ensure that the high type is indifferent over all stopping times that stop on $[t_0, t_0 + s]$, it must be that

$$
\frac{\partial W(x^h, t_0, \tau_q)}{\partial q} = 0
$$

for all $q \in [0, s]$. Letting $g = \rho + r$ and $f = g + \lambda$, this requirement can be written as,

$$
\begin{aligned}
0 &= \frac{\partial W(x^h, t_0, \tau_q)}{\partial q} \\
&= \frac{\partial}{\partial q}\left(\int_0^q \lambda e^{-\lambda t}\left[\int_0^t e^{-gs}(x^h - \rho\overline{p})ds - e^{-gt}p_{t_0+t}\right]dt + e^{-\lambda q}\left(\int_0^q e^{-gt}(x^h - \rho\overline{p})dt - e^{-gq}p_{t_0+q}\right)\right) \\
&= \frac{\partial}{\partial q}\left(\frac{x^h - \rho\overline{p}}{f}(1 - e^{-fq}) - \lambda\int_0^q e^{-ft}p_{t_0+t}dt - e^{-fq}p_{t_0+q}\right) \\
&= (x^h - \rho\overline{p})e^{-fq} - \lambda e^{-fq}p_{t_0+q} + fe^{-fq}p_{t_0+q} - e^{-fq}\frac{\partial p_{t_0+q}}{\partial q}
\end{aligned}
$$

$$= (x^h - \rho\overline{p})e^{-fq} + ge^{-fq}p_{t_0+q} - e^{-fq}\frac{\partial p_{t_0+q}}{\partial q}$$

The solution to this equation with initial condition $p_{t_0} = p$ is $p_{t_0+q} = \frac{x^h - \rho\overline{p}}{g}(e^{gq} - 1) + e^{gq}p$.[54] Rearranging leads to

$$t^I(p, p') = \frac{1}{g}ln\left(\frac{p' + \frac{x^h - \rho\overline{p}}{g}}{p + \frac{x^h - \rho\overline{p}}{g}}\right). \tag{G.3}$$

Guess solution to equation (G.2)**.** I propose that an optimal policy in equation (G.2) is

(G.4)

$$\begin{cases} \text{if } p < \frac{\rho\overline{p}}{\rho+r} - \frac{x^h}{\rho+r+\lambda}, & d(p) = 1, \quad p'(p) = \frac{\rho\overline{p}}{\rho+r} - \frac{x^h}{\rho+r+\lambda}, \quad t(p) = t^I(p, \frac{\rho\overline{p}}{\rho+r} - \frac{x^h}{\rho+r+\lambda}) \\ \text{if } p = \frac{\rho\overline{p}}{\rho+r} - \frac{x^h}{\rho+r+\lambda}, & d(p) = 0, \quad p'(p) = \frac{\rho\overline{p}}{\rho+r} - \frac{x^h}{\rho+r+\lambda}, \quad t(p) = \infty \end{cases}$$

Let $V^*_\gamma(p)$ be the value function associated to this policy. The second part of the guess can be immediately verified, since $t(p) = \infty$ and $d(p) = 0$ is the only feasible policy when $p = \frac{\rho\overline{p}}{\rho+r} - \frac{x^h}{\rho+r+\lambda}$, and the choice of $p'(p)$ in this case enters neither the regulator's value nor the incentive compatibility conditions. It remains to verify the first part of the guess.

Verification. To verify the first part of the guess, it is sufficient to consider one-shot deviations from the proposed optimal policy to policies with $d = 0$. Any deviation with $d = 1$ but $p' < p'(p)$, delivers the regulator exactly the same value as the guess.[55]

Recall that we let $f = \rho + r + \lambda$ and $g = \rho + r$. Let $p^{final} \equiv \frac{\rho\overline{p}}{\rho+r} - \frac{x^h}{\rho+r+\lambda}$. The regulator's value under the proposed guess is

$$V^*_\gamma(p) = -e^{-(\gamma+r)(t^I(p,p^{final}))}\int_0^\infty \left(e^{-(\gamma+r)t}\int_t^\infty e^{-f(s-t)}ds\right)dt$$
$$= -e^{-(\gamma+r)(t^I(p,p^{final}))}\frac{1}{f(\gamma+r)}$$

[54] It can be verified that p_{t_0+q} must be differentiable in q. One can proceed with only the knowledge that $\lambda\int_0^q e^{-ft}p_{t_0+q}dt + e^{-fq}p_{t_0+q}$ is differentiable, which follows immediately from differentiability of $W(x^h, t, \tau_q)$ (since it is constant on the interval) and $\frac{x^h - \rho\overline{p}}{f}(1 - e^{-fq})$, and arrive at the same conclusion.

[55] And generates the exact same policy path. This is a consequence of the fact that for any p, p'' and $p' \in [p, p'']$, $t^I(p, p'') = t^I(p, p') + t^I(p', p'')$.

Then, to verify the guess, I must show that:

(G.5)

$$\frac{-e^{-(\gamma+r)(t^I(p,p^{final}))}}{f(\gamma+r)} = \begin{cases} \sup\limits_{t\geq 0,p'} \quad \frac{-(1-e^{-(\gamma+r)t})}{(\gamma+r)(\rho+\lambda-\gamma)} + \frac{1-e^{-(\rho+\lambda+r)t}}{(\rho+\lambda-\gamma)f} - e^{-(r+\gamma)(t+t^I(p',p^{final}))}\frac{1}{f(\gamma+r)} \\ \text{subject to} \\ w^h(t) - e^{-(\rho+r)t}p' \leq -p \\ p' \in \mathcal{P} \end{cases}$$

Since $t^I(p', p^{final})$ is decreasing in p', the optimal choice of p' given t satisfies the incentive constraint at equality. Let $s(t,p) = t^I\left((w^h(t)+p)e^{(\rho+r)t}, p^{final}\right)$ and define

$$v^\gamma(t) \equiv \frac{-(1-e^{-(\gamma+r)t})}{(\gamma+r)(\rho+\lambda-\gamma)} + \frac{1-e^{-(\rho+\lambda+r)t}}{(\rho+\lambda-\gamma)f} - \frac{1}{f(\gamma+r)}e^{-(r+\gamma)(t+s(t,p))}$$

Plugging this into equation (G.5) and replacing the incentive constraint with a condition that guarantees that the choice of t can be paired with a feasible p' that satisfies the incentive constraint, the verification problem reduces to showing

$$\text{(G.6)} \qquad \frac{\overbrace{-e^{-(\gamma+r)s(0,p)}}^{v_0^\gamma \equiv}}{f(\gamma+r)} = \begin{cases} \sup\limits_{t\geq 0} \quad v^\gamma(t) \\ \text{subject to} \\ e^{(\rho+r)t}(w^h(t)+p) \in \mathcal{P} \end{cases}$$

Since $v^\gamma(0) = v_0^\gamma$, the verification will be complete if I can show that $\frac{\partial v^\gamma(t)}{\partial t} \leq 0$ on the set of t such that $e^{(\rho+r)t}(w^h(t)+p) \in \mathcal{P}$. To see why, first observe that the constraint in (G.6) generates a set of feasible $t \in \mathbb{R}_+$ that is of the form

$$\text{(G.7)} \qquad [0, t^{lower}] \cup [t^{upper}, \infty]$$

such that at t^{upper} and t^{lower}, the only feasible p' is $\frac{\rho\overline{p}}{g} - \frac{x^h}{f}$. Second, $\frac{\partial v^\gamma(t)}{\partial t} \leq 0$ implies that the solution is attained either at t^{upper} or at 0. That the solution is always attained at 0 can be shown by directly comparing the regulator's value at the two choices 0 and t^{upper}, and is postponed until after I have shown that $\frac{\partial v^\gamma(t)}{\partial t} \leq 0$ on the set of feasible t.

To show that $\frac{\partial v^\gamma(t)}{\partial t} \leq 0$, plug in the definition of t^I from equation (G.3) into $s(t,p)$,

$$\begin{aligned} s(t,p) &= \frac{1}{g} ln\left(\frac{x^h(\frac{1}{g}-\frac{1}{f})}{e^{gt}\left(w^h(t)+p\right)+\frac{x^h-\rho\overline{p}}{g}}\right) \\ &= \frac{1}{g} ln\left(\frac{x^h(\frac{1}{g}-\frac{1}{f})}{e^{gt}\left(\frac{1-e^{-ft}}{f}x^h - \frac{\rho\overline{p}}{g} + p\right)+\frac{x^h}{g}}\right) \end{aligned}$$

where I plugged in $w^h(t) = x^h \frac{1-e^{-ft}}{f} - \frac{\rho\overline{p}}{g}(1-e^{-gt})$. Then,

$$\begin{aligned}
v^\gamma(t) &= \frac{-(1-e^{-(\gamma+r)t})}{(\gamma+r)(\rho+\lambda-\gamma)} + \frac{1-e^{-ft}}{(\rho+\lambda-\gamma)f} - \frac{1}{f(\gamma+r)} e^{-(r+\gamma)(t+s(t,p))} \\
&= \frac{-(1-e^{-(\gamma+r)t})}{(\gamma+r)(\rho+\lambda-\gamma)} + \frac{1-e^{-ft}}{(\rho+\lambda-\gamma)f} - \frac{e^{-(r+\gamma)t}}{f(\gamma+r)} \left(\frac{e^{gt}\left(\frac{1-e^{-ft}}{f}x^h - \frac{\rho\overline{p}}{g} + p\right) + \frac{x^h}{g}}{x^h(\frac{1}{g} - \frac{1}{f})} \right)^{\frac{\gamma+r}{\rho+r}} \\
&= \frac{-(1-e^{-(\gamma+r)t})}{(\gamma+r)(\rho+\lambda-\gamma)} + \frac{1-e^{-ft}}{(\rho+\lambda-\gamma)f} - \frac{1}{f(\gamma+r)} \left(\frac{\frac{1-e^{-ft}}{f}x^h - \frac{\rho\overline{p}}{g} + p + e^{-gt}\frac{x^h}{g}}{x^h(\frac{1}{g} - \frac{1}{f})} \right)^{\frac{\gamma+r}{\rho+r}}
\end{aligned}$$

The last term in parentheses in the second line is always between 0 and 1 whenever the incentive constraint is satisfied (i.e. $e^{gt}(w^h(t)+p) \in \mathcal{P}$), since the denominator is the numerator evaluated at p^{final}. As a result, the last term in parentheses in the last line is smaller than $e^{-(\rho+r)t}$ for feasible choices of t (given p). Differentiating $v^\gamma(t)$,

$$\begin{aligned}
\frac{\partial v^\gamma(t)}{\partial t} &= \frac{1}{\rho+\lambda-\gamma}\left(e^{-ft} - e^{-(\gamma+r)t}\right) + \frac{e^{-gt} - e^{-ft}}{\lambda} \left(\frac{p + \frac{x^h e^{-gt}}{g} + \frac{1-e^{-ft}}{f}x^h - \frac{\rho\overline{p}}{g}}{\frac{x^h\lambda}{fg}} \right)^{\frac{\gamma-\rho}{\rho+r}} \\
&\leq \frac{1}{\rho+\lambda-\gamma}\left(e^{-ft} - e^{-(\gamma+r)t}\right) + \frac{1}{\lambda}\left(e^{-gt} - e^{-ft}\right) e^{-(\gamma-\rho)t}
\end{aligned}$$

where the second line follows from the fact that, as noted above, the last term in parentheses in the first line is smaller than e^{-gt}. Rearranging,

$$\begin{aligned}
\frac{\partial v^\gamma(t)}{\partial t} &\leq \frac{1}{\rho+\lambda-\gamma}\left(e^{-ft} - e^{-(\gamma+r)t}\right) + \frac{1}{\lambda}\left(e^{-gt} - e^{-ft}\right) e^{-(\gamma-\rho)t} \\
&= e^{-(\gamma+r)t}\left(\frac{1}{\lambda} - \frac{1}{\rho+\lambda-\gamma}\right) + e^{-ft}\left(\frac{1}{\rho+\lambda-\gamma} - \frac{e^{-(\gamma-\rho)t}}{\lambda}\right) \\
&= \frac{e^{-ft}}{\lambda}\left[\frac{\overbrace{\lambda(1-e^{-(\gamma-\rho)t}) + (\rho-\gamma)(e^{(\rho+\lambda-\gamma)t} - e^{(\rho-\gamma)t})}^{c(t)\equiv}}{\rho+\lambda-\gamma}\right] \qquad \text{(G.8)}
\end{aligned}$$

The term $c(t)$ is 0 at $t=0$ and differentiating yields,

$$\begin{aligned}
\frac{\partial c(t)}{\partial t} &= \frac{\lambda(\gamma-\rho)e^{-(\gamma-\rho)t} + (\rho-\gamma)^2 e^{-(\gamma-\rho)t}(e^{\lambda t}-1) + \lambda(\rho-\gamma)e^{(\rho+\lambda-\gamma)t}}{\rho+\lambda-\gamma} \\
&= \frac{(\gamma-\rho)e^{-(\gamma-\rho)t}\left[\lambda + (\gamma-\rho)[e^{\lambda t}-1] - e^{\lambda t}\lambda\right]}{\rho+\lambda-\gamma} \\
&= (\gamma-\rho)e^{-(\gamma-\rho)t}(1-e^{\lambda t}) \\
&\leq 0
\end{aligned}$$

where the last line is a result of the fact that $\gamma > \rho$ and $1 - e^{\lambda t} \leq 0$. Plugging into inequality (G.8) implies that

$$\frac{\partial v^{\gamma}(t)}{\partial t} \leq 0.$$

To complete the verification, I need to show that $v^{\gamma}(0) \geq v^{\gamma}(t^{upper})$, where t^{upper} is defined in (G.7). To see that, observe that in the proposed optimal policy, the regulator induces reporting by high types for a period of length

$$t^{I}(p, p^{final}) = \frac{1}{g} ln \left(\frac{p^{final} + \frac{x^h - \rho\bar{p}}{g}}{p + \frac{x^h - \rho\bar{p}}{g}} \right)$$

and then gets $-\frac{1}{f(\gamma+r)}$, so the regulator's payoff is

$$\begin{aligned} v^{\gamma}(0) &\equiv -\frac{1}{f(\gamma+r)} \left(\frac{p^{final} + \frac{x^h - \rho\bar{p}}{g}}{p + \frac{x^h - \rho\bar{p}}{g}} \right)^{-\frac{\gamma+r}{g}} \\ &= -\frac{1}{f(\gamma+r)} \left(\frac{p + \frac{x^h - \rho\bar{p}}{g}}{x^h(\frac{1}{g} - \frac{1}{f})} \right)^{\frac{\gamma+r}{g}} \end{aligned}$$

The alternative choice is $t = t^{upper}$ and $p' = \frac{\rho\bar{p}}{g} - \frac{x^h}{f}$ (the maximum feasible), with the incentive constraint satisfied at equality, i.e. $\frac{1-e^{-ft}}{f}x^h - \frac{\rho\bar{p}}{g}(1 - e^{-gt}) - (\frac{\rho\bar{p}}{g} - \frac{x^h}{f})e^{-gt} = -p$. The value under this alternative policy is (letting $t = t^{upper}$)

$$\begin{aligned} v^{\gamma}(t) &= -\frac{1 - e^{-(\gamma+r)t}}{(\gamma+r)(\rho+\lambda-\gamma)} + \frac{1 - e^{-ft}}{f(\rho+\lambda-\gamma)} - \frac{1}{f(\gamma+r)} e^{-(\gamma+r)t} \\ &= \frac{e^{-(\gamma+r)t} - e^{-ft}}{f(\rho+\lambda-\gamma)} - \frac{1}{(\gamma+r)f} \\ &= -\frac{1}{f(\gamma+r)} \left(1 - \frac{\gamma+r}{\rho+\lambda-\gamma} \left(e^{-(\gamma+r)t} - e^{-ft} \right) \right) \end{aligned} \tag{G.9}$$

Using the incentive constraint evaluated at $t = t^{upper}$ to substitute for p in $v^{\gamma}(0)$ yields

$$\begin{aligned} v^{\gamma}(0) &= -\frac{1}{f(\gamma+r)} \left(\frac{x^h(\frac{1}{g} - \frac{1}{f}) - \frac{x^h}{f}(e^{-gt} - e^{-ft})}{x^h(\frac{1}{g} - \frac{1}{f})} \right)^{\frac{\gamma+r}{g}} \\ &= -\frac{1}{f(\gamma+r)} \left(1 - \frac{g}{\lambda}(e^{-gt} - e^{-ft}) \right)^{\frac{\gamma+r}{g}} \end{aligned} \tag{G.10}$$

Now let,

$$\hat{v}(x) \equiv -\frac{1}{f(\gamma+r)} \left[\left(1 + \frac{x}{f-x} \left(e^{-ft} - e^{-xt} \right) \right)^{\frac{1}{x}} \right]^{\gamma+r}$$

and observe that $v^\gamma(0) = \hat{v}(g)$ and $v^\gamma(t) = \hat{v}(\gamma + r)$. Since, $g < \gamma + r$, the proof will be complete if I show that $\hat{v}(x)$ is decreasing in x, because then $v^\gamma(0) \leq v^\gamma(t)$. To this end, I will show

$$\frac{\partial}{\partial x}\left[\left(1 + \frac{x}{f-x}\left(e^{-ft} - e^{-xt}\right)\right)^{\frac{1}{x}}\right] \geq 0$$

This is relegated to Lemma G.1 below. This completes the verification, so $V_\gamma^*(p)$ solves equation (G.2).

Conclusion. Given the solution to the recursive representation $V_\gamma^*(p)$, and observing that $V_\gamma^*(p)$ is maximized at $p = \underline{p} = 0$, we have

$$V_\gamma^* = \max_{t_0} \int_0^{t_0} e^{-(\gamma+r)s}\left(-\int_s^{t_0} e^{-(\rho+r+\lambda)(q-s)}dq\right) ds + e^{-rt_0}V_\gamma^*(\underline{p}).$$

Applying the policy functions associated with $V_\gamma^*(.)$ then generates the optimal path described in the theorem. □

Lemma G.1. *For any* $t, x \geq 0$

$$\frac{\partial}{\partial x}\left[\left(1 + \frac{x}{f-x}(e^{-ft} - e^{-xt})\right)^{\frac{1}{x}}\right] \geq 0$$

Proof. Observe that any function $g(x)$ such that $g(x) = e^{-xt}h(x,t)$ has the property that

$$(g(x))^{\frac{1}{x}} = e^{-t}h(x,t)^{\frac{1}{x}}$$

is increasing if $h(x,t)$ is increasing in x. As a result, it is sufficient to show that

$$h(x,t) \equiv \frac{1 + \frac{x}{f-x}(e^{-ft} - e^{-xt})}{e^{-xt}} = e^{xt} + \frac{x}{f-x}(e^{(x-f)t} - 1)$$

is weakly increasing in x. To this end write

$$\frac{\partial h(x,t)}{\partial x} = te^{xt} + \frac{tx}{f-x}(e^{(x-f)t}) + \frac{e^{(x-f)t} - 1}{f-x} + \frac{x}{(f-x)^2}(e^{(x-f)t} - 1).$$

At $t = 0$, $\frac{\partial h(x,t)}{\partial x} = 0$, so it is sufficient to show that $\frac{\partial^2 h(x,t)}{\partial x \partial t} \geq 0$. Observe

$$\begin{aligned}\frac{\partial^2 h(x,t)}{\partial x \partial t} &= e^{xt} + xte^{xt} + \frac{xe^{(x-f)t}}{f-x} - txe^{(x-f)t} - e^{(x-f)t} + \frac{xe^{(x-f)t}}{x-f} \\ &= e^{xt}(1+xt)(1-e^{-ft}) \geq 0\end{aligned}$$

which completes the proof. □